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The standing vortex behind a disk normal to uniform flow at small Reynolds number

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Viscous incompressible flow around a circular disk set normal to undisturbed flow is theoretically studied at small values of the Reynolds number. The critical Reynolds number at which a standing vortex first appears behind the disk is zero, and the flow separates from the edge of the disk. Several other features of the flow are obtained from the present analysis.

1. Introduction

The critical Reynolds number at which a standing vortex first appears behind a body may be one of the interesting parameters in the dynamics of laminar viscous flow. In the present paper, we concern ourselves with the critical Reynolds number for a circular disk placed perpendicular to uniform viscous incompressible fluid flow. In order to attain this objective, we must have an exact expression for the Stokes stream function at small values of the Reynolds number R. Thus the present problem must include the use of the technique of so-called matched asymptotic expansions for this axisymmetric flow. Using this technique, Breach (1961) has given drag force experienced by all ellipsoids of revolution, both prolate and oblate, but it appears that there is no explicit expression for the stream function. On the other hand, the similar problem for a flat plate has been discussed by Miyagi (1978), who reported that the critical Reynolds number is zero and the flow separates from the edge of the plate. In the two-dimensional flow, however, it may be supposed that the matching procedures are not necessarily complete.

Here, we seek for an exact expression up to the second approximation in the expansions of the stream function $\psi = \psi_0 + R\psi_1 + \dots$. By evaluating the stream function thus obtained and investigating the behaviour of the vorticity in the vicinity of the disk edge, it is revealed that the critical Reynolds number under discussion is zero and the separated streamline arises from the edge of the disk. Furthermore, the vorticity distribution on the disk, the separation angle of the flow at the edge of the disk and the size of the standing vortex are discussed, and an example of flow pattern near the disk is also drawn at the value of $R = 0.3$.

2. Fundamental equations and boundary conditions

We consider steady viscous incompressible fluid flow past a circular disk set normal to the uniform flow of speed U. Since the flow field is axisymmetric, we take cylindrical coordinates (x, y) normalized by the radius a of the disk. The x-axis coincides with the symmetry axis, the y-axis is normal to it and the disk is placed on the y-axis between $y = 0$ and 1 $(x = 0)$.

Introducing the Stokes stream function defined as

$$q_x = \frac{1}{y}\frac{\partial \psi}{\partial y}, \quad q_y = -\frac{1}{y}\frac{\partial \psi}{\partial x}, \tag{2.1}$$

where (Uq_x, Uq_y) denote the (x, y)-components of the fluid velocity, the equation of continuity is identically satisfied, and the Navier–Stokes equations of motion become

$$\left.\begin{aligned} \mathrm{E}^4\psi &= -Ry\frac{\partial(\psi, y^{-2}\mathrm{E}^2\psi)}{\partial(x,y)}, \\ \mathrm{E}^2 &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y}, \end{aligned}\right\} \tag{2.2}$$

where

$R = Ua/\nu$ is the Reynolds number and ν the kinematic viscosity of the fluid. The vorticity ω is also given by

$$\omega = \frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y} = -\frac{1}{y}\mathrm{E}^2\psi. \tag{2.3}$$

Then, for the sake of convenience in the treatment of the disk, we adopt oblate spheroidal coordinates (λ, ζ) in place of (x, y) as

$$\left.\begin{aligned} x = \lambda\zeta, \quad y = (1+\lambda^2)^{\frac{1}{2}}(1-\zeta^2)^{\frac{1}{2}}, \\ \infty > \lambda \geqslant 0, \quad 1 \geqslant \zeta \geqslant -1. \end{aligned}\right\} \tag{2.4}$$

$\lambda = 0$ gives the surface of the disk, $\lambda \to \infty$ corresponds to infinity, and $\zeta = 1$ and $\zeta = -1$ to the down- and upstream sides of the symmetric axis respectively. Furthermore, the differential operators are transformed into

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{\lambda^2+\zeta^2}\left[\zeta(1+\lambda^2)\frac{\partial}{\partial \lambda} + \lambda(1-\zeta^2)\frac{\partial}{\partial \zeta}\right], \\ \frac{\partial}{\partial y} &= \frac{1}{\lambda^2+\zeta^2}\left[\lambda\frac{\partial}{\partial \lambda} - \zeta\frac{\partial}{\partial \zeta}\right], \\ \mathrm{E}^2 &= \frac{1}{\lambda^2+\zeta^2}\left[(1+\lambda^2)\frac{\partial^2}{\partial \lambda^2} + (1-\zeta^2)\frac{\partial^2}{\partial \zeta^2}\right]. \end{aligned}\right\} \tag{2.5}$$

The boundary conditions are written in terms of (λ, ζ) as

$$\left.\begin{aligned} \psi = 0 \quad &\text{on} \quad \zeta = \pm 1, \\ \psi = 0, \quad \frac{\partial \psi}{\partial \lambda} = 0 \quad &\text{at} \quad \lambda = 0, \\ \psi \to \tfrac{1}{2}(1-\zeta^2)\lambda^2 \quad &\text{as} \quad \lambda \to \infty. \end{aligned}\right\} \tag{2.6}$$

3. First approximations

In the present problem, it may be sufficient to assume the inner solution ψ in the form

$$\psi = \psi_0 + R\psi_1 + \ldots, \tag{3.1}$$

under the condition $R \ll 1$. Hence we have the equation for the first approximation ψ_0 in the inner region as

$$\mathrm{E}^4\psi_0 = 0. \tag{3.2}$$

The solution of (3.2) subject to (2.6) can be seen in some available textbooks (e.g. Happel & Brenner 1973), that is

$$\psi_0 = \frac{1}{\pi}(1-\zeta^2)\,[(1+\lambda^2)\tan^{-1}\lambda - \lambda]. \tag{3.3}$$

When $\lambda \gg 1$,

$$\psi_0 \sim \tfrac{1}{2}(1-\zeta^2)\left(\lambda^2 - \frac{4}{\pi}\lambda + \dots\right). \tag{3.4}$$

On the other hand, we must consider the outer region as usual in the procedures of matched asymptotic expansions. We take there cylindrical coordinates (X, Y) and the outer stream function Ψ defined as

$$X = Rx, \quad Y = Ry, \quad \Psi = R^2\psi, \tag{3.5}$$

in terms of which the governing equation (2.2) becomes

$$\left.\begin{aligned} \mathrm{E}_0^4\,\Psi &= -Y\frac{\partial(\Psi, Y^{-2}\mathrm{E}_0^2\,\Psi)}{\partial(X, Y)}, \\ \text{where}\qquad \mathrm{E}_0^2 &\equiv \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} - \frac{1}{Y}\frac{\partial}{\partial Y}. \end{aligned}\right\} \tag{3.6}$$

Again, Ψ may be assumed to be of the same type as ψ in expansion as

$$\Psi = \Psi_0 + R\Psi_1 + \dots. \tag{3.7}$$

Substituting (3.7) into (3.6), the differential equation for Ψ_0 is given by

$$\mathrm{E}_0^4\,\Psi_0 = -Y\frac{\partial(\Psi_0, Y^{-2}\mathrm{E}_0^2\,\Psi_0)}{\partial(X, Y)}. \tag{3.8}$$

Within the present approximation, it can be easily proved that the usual spherical coordinates referred to the origin $r = (x^2+y^2)^{\frac{1}{2}}, \mu = \cos\theta = x/r$ have the relations $r \approx \lambda$ and $\mu \approx \zeta$. Putting $\rho = Rr \approx R\lambda$, (3.4) can be transformed in terms of outer quantities (ρ, μ) as follows:

$$\Psi_0 = R^2\psi_0 \sim \tfrac{1}{2}(1-\mu^2)\left(\rho^2 - \frac{4}{\pi}\rho R + \dots\right). \tag{3.9}$$

Hence we get

$$\Psi_0 = \tfrac{1}{2}(1-\mu^2)\,\rho^2. \tag{3.10}$$

Noting that $\mathrm{E}_0^2\,\Psi_0 = \partial\Psi_0/\partial X = 0$, it is readily seen that (3.10) is the solution of (3.8) and constitutes the uniform flow at infinity.

4. Second approximations

By substitution of (3.10) into (3.2), the differential equation for Ψ_1 can be reduced to

$$\mathrm{E}_0^2\left(\mathrm{E}_0^2 - \frac{\partial}{\partial X}\right)\Psi_1 = 0, \tag{4.1}$$

making use of $\partial\Psi_0/\partial Y = \rho(1-\mu^2)^{\frac{1}{2}}$. In a similar manner to the case of a sphere treated by Proudman & Pearson (1957), the solution of (4.1) can be represented by the Oseenlet in the form:

$$\Psi_1 = -\frac{4}{\pi}(1+\mu)\,[1-\mathrm{e}^{-\frac{1}{2}\rho(1-\mu)}]. \tag{4.2}$$

The numerical factor $-4/\pi$ has been determined from the matching the second term of ψ_0 in (3.9) with the first term of the expansion of (4.2) for $\rho \ll 1$, that is

$$\Psi_1 \sim -\frac{2}{\pi}(1-\mu^2)\rho[1-\tfrac{1}{4}\rho+\tfrac{1}{4}\rho\mu+\ldots]. \tag{4.3}$$

Next, we proceed to the inner solution ψ_1, whose governing equation can be transformed from (2.2) with (3.1) and (3.3) by some cumbersome but straightforward manipulations to

$$\mathrm{E}^4\psi_1 = \frac{8}{\pi^2}\frac{\zeta(1-\zeta^2)}{(\lambda^2+\zeta^2)^3}\left[(\lambda^2+\zeta^2)\left\{\lambda-\frac{2\lambda}{1+\lambda^2}+(1+\lambda^2)\tan^{-1}\lambda\right\}\right.$$
$$\left.+2\lambda^2\{\lambda-2(1+\lambda^2)\tan^{-1}\lambda\}\right]. \tag{4.4}$$

The particular integral ψ_{P} of the above can be obtained stepwise as follows:

$$\mathrm{E}^2\psi_{\mathrm{P}} = \frac{4}{\pi^2}\left[\frac{\zeta(1-\zeta^2)}{\lambda^2+\zeta^2}(\lambda^2\tan^{-1}\lambda+2\tan^{-1}\lambda-2\lambda)-\tan^{-1}\frac{\zeta}{\lambda}-\zeta\tan^{-1}\lambda\right], \tag{4.5}$$

and

$$\psi_{\mathrm{P}} = -\frac{1}{\pi^2}\zeta(1-\zeta^2)[2\lambda(1+\lambda^2)(\tan^{-1}\lambda)^2+5(1+\lambda^2)\tan^{-1}\lambda+\lambda]$$
$$-\frac{2}{\pi^2}\left[(\lambda^2-\zeta^2)\tan^{-1}\frac{\zeta}{\lambda}+\zeta(\lambda^2-1)\tan^{-1}\lambda+\lambda\zeta\ln\frac{\lambda^2+\zeta^2}{1+\lambda^2}\right]. \tag{4.6}$$

It should be noted that the expression for ψ_{P} for $\lambda \gg 1$ is

$$\psi_{\mathrm{P}} \sim -\frac{1}{\pi^2}\zeta(1-\zeta^2)(\tfrac{1}{2}\pi^2\lambda^3+\tfrac{1}{2}\pi\lambda^2)-\frac{1}{\pi}\zeta\lambda^2+O(\lambda), \tag{4.7}$$

and the second term in the bracket of (4.7) is automatically matched with the third term in (4.3).

Furthermore, in order to satisfy the boundary conditions (2.6), we must add some solutions ψ_{C} of the homogeneous equation for (4.4) such that

$$\psi_{\mathrm{C}} = \frac{1}{\pi^2}(1-\zeta^2)[(1+\lambda^2)\tan^{-1}\lambda-\lambda]+\frac{6}{\pi^2}\zeta(1-\zeta^2)\lambda(1+\lambda^2)$$
$$+\left(\frac{1}{2}-\frac{6}{\pi^2}\right)\frac{2}{\pi}\zeta(1-\zeta^2)[\lambda^2+\lambda(1+\lambda^2)\tan^{-1}\lambda]+\frac{1}{\pi}\zeta(\lambda^2-1), \tag{4.8}$$

where use was made of the fact that if $\mathrm{E}^2\chi(\lambda,\zeta)=0$, $\mathrm{E}^4[\lambda\zeta\chi(\lambda,\zeta)]=0$, and the numerical factor $1/\pi^2$ in the first term has been settled by matching the second term of (4.3) with the first term of the outer expression for this term. Moreover, the term $O(\lambda^3)$ in (4.7) is cancelled with that arising from (4.8).

In spite of the above considerations, however, our solutions $\psi_{\mathrm{P}}+\psi_{\mathrm{C}}$ do not as yet satisfy the boundary conditions completely, and the remnants are written as

$$\left.\begin{aligned} \psi_{\mathrm{P}}+\psi_{\mathrm{C}}|_{\lambda=0} &= \frac{1}{\pi}(\zeta^2\,\mathrm{sgn}\,\zeta-\zeta), \\ \frac{\partial}{\partial\lambda}(\psi_{\mathrm{P}}+\psi_{\mathrm{C}})\bigg|_{\lambda=0} &= -\frac{2}{\pi^2}\zeta\ln\zeta^2. \end{aligned}\right\} \tag{4.9}$$

5. Additive solutions

These boundary conditions left behind mainly originate from the terms in (4.6), which are not of the type of separation of variables. However, there seems to be no obvious way to find such solutions of the homogeneous equation in closed form. Therefore we seek for a solution ψ_S represented by separation variables, which is assumed to satisfy the above conditions (4.9) in the following form:

$$\psi_S = F(\lambda, \zeta) + \lambda\zeta G(\lambda, \zeta), \tag{5.1}$$

where $E^2F(\lambda, \zeta) = E^2G(\lambda, \zeta) = 0$. It is clear from (4.9) that ψ_S should be an odd function in ζ. Then, if $E^2\chi(\lambda, \zeta) = 0$ and $\chi(\lambda, \zeta) = \Lambda(\lambda) Z(\zeta)$, we get two equations to determine $\Lambda(\lambda)$ and $Z(\zeta)$ as

$$\left.\begin{aligned} (1-\zeta^2) Z'' + CZ = 0, \\ (1+\lambda^2) \Lambda'' - C\Lambda = 0, \end{aligned}\right\} \tag{5.2}$$

where primes indicate differentiation with respect to the appropriate variable, and C is a separation constant. Putting $C = n(n+1)$, we can determine $Z_n(\zeta)$ under the consideration of the boundary conditions for ζ in the following form:

$$Z_n(\zeta) = (1-\zeta^2) P_n'(\zeta), \tag{5.3}$$

where $P_n(\zeta)$ is the Legendre polynomial of order n. Further, it can be easily proved that $Z_n(\zeta)$ has the following orthogonality property:

$$\int_{-1}^{1} \frac{Z_n(\zeta) Z_m(\zeta)}{1-\zeta^2} d\zeta = \begin{cases} \dfrac{2n(n+1)}{2n+1} & (m = n) \\ 0 & (m \neq n). \end{cases} \tag{5.4}$$

In a similar way, $\Lambda_n(\lambda)$ can be determined as

$$\Lambda_n(\lambda) = (1+\lambda^2) q_n'(\lambda), \tag{5.5}$$

where $q_n(\lambda)$ is the same function as that studied by Lamb (1932) and comes from the Legendre function of second kind, so that $\Lambda_n(\lambda)$ tends to zero for $\lambda \gg 1$. Thus we can write down the following recurrence relations for both $Z_n(\zeta)$ and $\Lambda_n(\lambda)$ from those of the Legendre function as

$$\left.\begin{aligned} Z_n(\zeta) &= \frac{1}{n-1} [(2n-1) \zeta Z_{n-1}(\zeta) - nZ_{n-2}(\zeta)], \\ \Lambda_n(\lambda) &= \frac{1}{n-1} [n\Lambda_{n-2}(\lambda) - (2n-1) \lambda \Lambda_{n-1}(\lambda)]. \end{aligned}\right\} \tag{5.6}$$

For conciseness, we show here the explicit forms of only the first four terms of both functions:

$$\left.\begin{aligned} Z_0 &= 0, & \Lambda_0 &= -1, \\ Z_1 &= 1-\zeta^2, & \Lambda_1 &= \lambda - (1+\lambda^2) \cot^{-1} \lambda, \\ Z_2 &= 3\zeta(1-\zeta^2), & \Lambda_2 &= 3\lambda(1+\lambda^2) \cot^{-1} \lambda - 3\lambda^2 - 2, \\ Z_3 &= \tfrac{3}{2}(1-\zeta^2)(5\zeta^2-1), & \Lambda_3 &= \tfrac{1}{2}[15\lambda^3 + 13\lambda - 3(5\lambda^2+1)(1+\lambda^2) \cot^{-1} \lambda]. \end{aligned}\right\} \tag{5.7}$$

Since $Z_{2n}(\zeta)$ and $Z_{2n+1}(\zeta)$ are odd and even in ζ respectively, $F(\lambda, \zeta)$ and $G(\lambda, \zeta)$ should have the following forms:

$$\left.\begin{aligned} F(\lambda,\zeta) &= \sum_{n=1}^{\infty} A_{2n} \Lambda_{2n}(\lambda)\, Z_{2n}(\zeta), \\ G(\lambda,\zeta) &= \sum_{m=0}^{\infty} B_{2m+1} \Lambda_{2m+1}(\lambda)\, Z_{2m+1}(\zeta), \end{aligned}\right\} \tag{5.8}$$

where A_{2n} and B_{2m+1} are constants to be determined in the following.

Combining (5.1) with (4.9) and (5.8), we have

$$\left.\begin{aligned} \frac{1}{\pi}(\zeta-\zeta^2 \operatorname{sgn}\zeta) = F(0,\zeta) &= \sum_{n=1}^{\infty} A_{2n} \Lambda_{2n}(0)\, Z_{2n}(\zeta), \\ \frac{2}{\pi^2}\zeta \ln \zeta^2 - \left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=0} &= \zeta \sum_{m=0}^{\infty} B_{2m+1} \Lambda_{2m+1}(0)\, Z_{2m+1}(\zeta). \end{aligned}\right\} \tag{5.9}$$

By multiplying both sides of the first equation in (5.9) by $Z_{2n}(\zeta)/(1-\zeta^2)$, and using the orthogonality (5.4), we obtain the equation for A_{2n} as

$$A_{2n} = \frac{4n+1}{4n(2n+1)} \frac{1}{\Lambda_{2n}(0)} \frac{1}{\pi} \int_{-1}^{1} \frac{\zeta-\zeta^2 \operatorname{sgn}\zeta}{1-\zeta^2} Z_{2n}(\zeta)\, d\zeta. \tag{5.10}$$

Applying the formulae for the Legendre function to the above integral, we arrive at

$$\int_{-1}^{1} \frac{\zeta-\zeta^2 \operatorname{sgn}\zeta}{1-\zeta^2} Z_{2n}(\zeta)\, d\zeta = (-1)^{n+1} \frac{4}{(2n+2)(2n-1)} \frac{(2n)!}{(2^n n!)^2}. \tag{5.11}$$

From the recurrence relation (5.6), we get

$$\Lambda_{2n}(0) = -\frac{(2^n n!)^2}{(2n)!}. \tag{5.12}$$

Thus we can write A_{2n} in the following explicit form:

$$A_{2n} = (-1)^n \frac{2}{\pi} \frac{4n+1}{(2n+2)(2n+1)(2n)(2n-1)} \left[\frac{(2n)!}{(2^n n!)^2}\right]^2. \tag{5.13}$$

In the same way as for A_{2n}, we can obtain an expression for B_{2m+1} as

$$B_{2m+1} = \frac{4m+3}{(4m+4)(2m+1)} \frac{1}{\Lambda_{2m+1}(0)} \int_{-1}^{1} \left[\frac{2}{\pi^2}\zeta \ln \zeta^2 - \left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=0}\right] \frac{Z_{2m+1}(\zeta)}{\zeta(1-\zeta^2)}\, d\zeta, \tag{5.14}$$

where

$$\left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=0} = \sum_{n=1}^{\infty} A_{2n} \Lambda'_{2n}(0)\, Z_{2n}(\zeta). \tag{5.15}$$

Using the formula

$$(1+\lambda^2)\Lambda'_n(\lambda) = -\frac{n(n+1)}{2n+1}[\Lambda_{n-1}(\lambda)+\Lambda_{n+1}(\lambda)], \tag{5.16}$$

which can be transformed from the corresponding one for the Legendre function, we have

$$\Lambda'_{2n}(0) = \frac{\pi}{2}(2n)(2n+1)\frac{(2n)!}{(2^n n!)^2}, \tag{5.17}$$

so that (5.15) can be rewritten in the form

$$\left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=0} = \sum_{n=1}^{\infty} (-1)^n \frac{4n+1}{(2n+2)(2n-1)}\left[\frac{(2n)!}{(2^n n!)^2}\right]^3 Z_{2n}(\zeta). \tag{5.18}$$

It should be remembered that the behaviour of the above coefficient for $n \gg 1$ can be expressed by

$$(-1)^n \frac{4n+1}{(2n+2)(2n-1)}\left[\frac{(2n)!}{(2^n n!)^2}\right]^3 \sim (-1)^n \pi^{-\frac{3}{2}} n^{-\frac{5}{2}}, \tag{5.19}$$

taking into account Stirling's formula $n! \sim (2\pi n)^{\frac{1}{2}} n^n \mathrm{e}^{-n}$. On the other hand, although the logarithmic term in the integral (5.14) can be estimated for various $Z_{2m+1}(\zeta)$, it may be shown that the replacement

$$\frac{2}{\pi^2}\zeta \ln \zeta^2 = \sum_{n=1}^{\infty} C_{2n} Z_{2n}(\zeta) \tag{5.20}$$

is satisfactory with respect to convergence of the series. Using the recurrence relation for the Legendre function,

$$P'_{2n}(\zeta) = \frac{1}{4n+1}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}[P_{2n+1}(\zeta) - P_{2n-1}(\zeta)], \tag{5.21}$$

and (5.4), we can show that

$$C_{2n} = (-1)^n \frac{4}{\pi^2}\frac{4n+1}{(2n+1)^2(2n)^2}\frac{(2^n n!)^2}{(2n)!}. \tag{5.22}$$

It is readily seen that the behaviour of C_{2n} for $n \gg 1$ coincides completely with (5.18); therefore the convergence of the term in the square bracket in (5.14) is greatly improved. Again, since the values of the term $A_{2m+1}(0)$ are estimated as

$$A_{2m+1}(0) = -\frac{\pi}{2}(2m+1)\frac{(2m)!}{(2^m m!)^2}, \tag{5.23}$$

from the recurrence relation, the last problem is the integration

$$I_{2m+1,\,2n} \equiv \int_{-1}^{1} \frac{Z_{2m+1}(\zeta)\, Z_{2n}(\zeta)}{\zeta(1-\zeta^2)}\,\mathrm{d}\zeta. \tag{5.24}$$

After some calculations, we can prove that

$$I_{2m+1,\,2n} = \begin{cases} (-1)^{m+n+1}\, 2\,\dfrac{(2m+1)!}{(2^m m!)^2}\dfrac{(2^n n!)^2}{(2n)!} & (2m+1 < 2n), \\ 0 & (2m+1 > 2n). \end{cases} \tag{5.25}$$

Hence we finally obtain the result for B_{2m+1} as

$$B_{2m+1} = (-1)^m \frac{2}{\pi}\frac{4m+3}{(2m+2)(2m+1)}\sum_{n=m+1}^{\infty}(4n+1)$$
$$\times\left[\left\{\frac{2}{\pi}\frac{1}{(2n+1)(2n)}\frac{(2^n n!)^2}{(2n)!}\right\}^2 - \frac{1}{(2n+2)(2n-1)}\left\{\frac{(2n)!}{(2^n n!)^2}\right\}^2\right]. \tag{5.26}$$

We show here numerical values of only the first three terms for A_{2n} and B_{2m+1}:

$$\left.\begin{aligned} A_2 &= -0.033157\ldots, & A_4 &= 0.002238\ldots, & A_6 &= -0.000481\ldots, \\ B_1 &= -0.090446\ldots, & B_3 &= 0.002738\ldots, & B_5 &= -0.000445\ldots. \end{aligned}\right\} \tag{5.27}$$

Gathering the results obtained above, we can write down the exact expression for the stream function $O(R)$ in the form

$$\begin{aligned}
\psi_1 = &-\frac{1}{\pi^2}\zeta(1-\zeta^2)\,[2\lambda(1+\lambda^2)(\tan^{-1}\lambda)^2+5(1+\lambda^2)\tan^{-1}\lambda+\lambda] \\
&-\frac{2}{\pi^2}\left[(\lambda^2-\zeta^2)\tan^{-1}\frac{\zeta}{\lambda}+\zeta(\lambda^2-1)\tan^{-1}\lambda+\lambda\zeta\ln\frac{\lambda^2+\zeta^2}{1+\lambda^2}\right] \\
&+\frac{1}{\pi^2}(1-\zeta^2)\,[(1+\lambda^2)\tan^{-1}\lambda-\lambda]+\frac{6}{\pi^2}\zeta(1-\zeta^2)\,\lambda(1+\lambda^2) \\
&+\left(\frac{1}{2}-\frac{6}{\pi^2}\right)\frac{2}{\pi}\zeta(1-\zeta^2)\,[\lambda^2+\lambda(1+\lambda^2)\tan^{-1}\lambda]+\frac{1}{\pi}\zeta(\lambda^2-1) \\
&+\sum_{n=1}^{\infty}A_{2n}\,\Lambda_{2n}(\lambda)\,Z_{2n}(\zeta)+\lambda\zeta\sum_{m=0}^{\infty}B_{2m+1}\,\Lambda_{2m+1}(\lambda)\,Z_{2m+1}(\zeta). \qquad (5.28)
\end{aligned}$$

The vorticity $\omega = \omega_0+R\omega_1+\dots$ associated with the above stream function can be readily obtained from the definition (2.3) as

$$\left.\begin{aligned}
\omega_0 = &-\frac{4}{\pi}[(1+\lambda^2)^{\frac{1}{2}}(1-\zeta^2)^{\frac{1}{2}}(\lambda^2+\zeta^2)]^{-1}\,\lambda(1-\zeta^2), \\
\omega_1 = &-\frac{4}{\pi^2}[(1+\lambda^2)^{\frac{1}{2}}(1-\zeta^2)^{\frac{1}{2}}(\lambda^2+\zeta^2)]^{-1}\Big[\lambda(1-\zeta^2)+\zeta(1-\zeta^2)\Big(\lambda^2\tan^{-1}\lambda+2\tan^{-1}\lambda \\
&-2\lambda+\pi-\frac{12}{\pi}\Big)-(\lambda^2+\zeta^2)\left(\tan^{-1}\frac{\zeta}{\lambda}+\zeta\tan^{-1}\lambda\right)+\tfrac{1}{2}\pi\zeta(1+\lambda^2) \\
&+\tfrac{1}{2}\pi^2\sum_{m=0}^{\infty}B_{2m+1}\{\zeta(1+\lambda^2)\,\Lambda'_{2m+1}(\lambda)\,Z_{2m+1}(\zeta)+\lambda(1-\zeta^2)\,\Lambda_{2m+1}(\lambda)\,Z'_{2m+1}(\zeta)\}\Big].
\end{aligned}\right\} \qquad (5.29)$$

6. Various results

In order to examine the behaviour of ω near the edge, we take polar coordinates (r,θ) with the origin at the edge of the disk and the axis $\theta = 0$ on $\zeta = 0$ $(\lambda > 0)$ such that

$$x = r\sin\theta = \lambda\zeta, \quad y = 1+r\cos\theta = (1+\lambda^2)^{\frac{1}{2}}(1-\zeta^2)^{\frac{1}{2}}. \qquad (6.1)$$

When $r \ll 1$ we have $\lambda \ll 1$, $\zeta \ll 1$ and the following relations:

$$\lambda \approx (2r)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta, \quad \zeta \approx (2r)^{\frac{1}{2}}\sin\tfrac{1}{2}\theta. \qquad (6.2)$$

Both λ and ζ are $O(r^{\frac{1}{2}})$. Hence order estimation of (5.29) yields the most-dominant terms $O(r^{-\frac{1}{2}})$ in the form

$$\left.\begin{aligned}
&\omega \sim \frac{A\lambda}{\lambda^2+\zeta^2}+\frac{B\zeta}{\lambda^2+\zeta^2}R \approx \frac{A\cos\frac{1}{2}\theta}{(2r)^{\frac{1}{2}}}+\frac{B\sin\frac{1}{2}\theta}{(2r)^{\frac{1}{2}}}R, \\
&\text{where} \\
&A = -\frac{4}{\pi}\left(1+\frac{R}{\pi}\right), \\
&B = -\left[\frac{6}{\pi}\left(1-\frac{8}{\pi^2}\right)+2\sum_{m=0}^{\infty}B_{2m+1}\,\Lambda'_{2m+1}(0)\,Z_{2m+1}(0)\right] \approx 0.1538\dots
\end{aligned}\right\} \qquad (6.3)$$

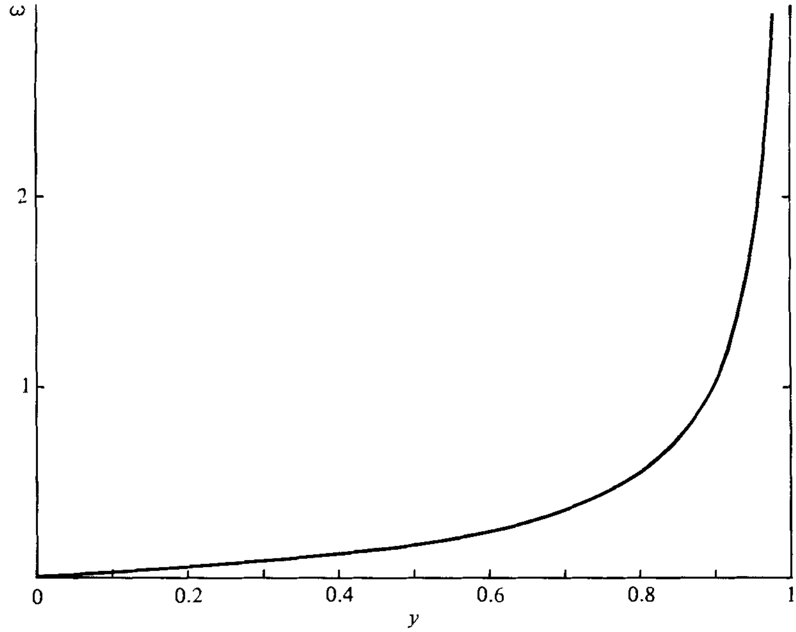

FIGURE 1. Vorticity distribution on the back face of the disk.

On the surface of the disk ($\theta = \mp\pi$), therefore, the second term of (6.3) is more dominant than the first, and this singular behaviour at the edge is entirely the same as in the case of a flat plate. Thus the vorticity near the edge $\omega \sim \omega_1 \sim \mp B/(2r)^{\frac{1}{2}}$ on the front ($\theta = -\pi$) and back ($\theta = \pi$) faces of the disk respectively.

On the other hand, we can easily evaluate the distribution of ω on the back face of the disk from (5.29), as shown in figure 1. Hence we may conclude that the standing vortex appears at any small but finite value of the Reynolds number, the flow separates from the edge, and the reattachment onto the disk does not occur. Thus it is shown that the conjectured equivalence to the case of a flat plate is valid.

In the vicinity of the edge, we can also calculate the zero-vorticity line and obtain the angle ϕ_ω between this line and the back face of the disk as

$$\phi_\omega = \pi - \theta_\omega = \tfrac{1}{4}\pi BR \approx 13.8°R. \tag{6.4}$$

Then, substituting (6.3) into (2.3), it is readily seen that the zero-streamline near the edge is governed by Poisson's equation, which can be easily integrated under the condition (2.6), giving

$$\psi \approx \tfrac{1}{2}\lambda^2(\tfrac{1}{3}A\lambda - B\zeta R). \tag{6.5}$$

Thus the separation angle of the flow is

$$\phi = \pi - \theta = \tfrac{3}{4}\pi BR \approx 41.5°R. \tag{6.6}$$

It may be noted that the separation angle is just 3 times ϕ_ω.

The size s of the standing vortex can be estimated from the fact that $q_x = 0$ on the line of symmetry, as indicated in figure 2. It may be seen from this figure that $s \approx 0.4R$ when $R \ll 1$.

In figure 3 an example of the flow pattern around the disk is shown at the value of $R = 0.3$.

Finally, it is easily proved that the drag coefficient in the present approximation is given by $C_D = (16/R)(1 + R/\pi)$, remembering that the only even functions of ζ in ψ are (3.3) and the third term in (5.28), and coincides with the result given by Breach (1961).

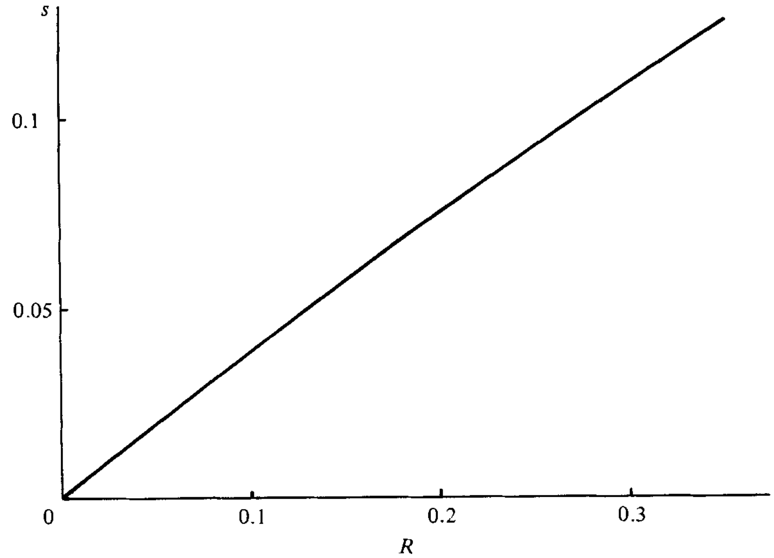

FIGURE 2. Size of standing vortex versus Reynolds number.

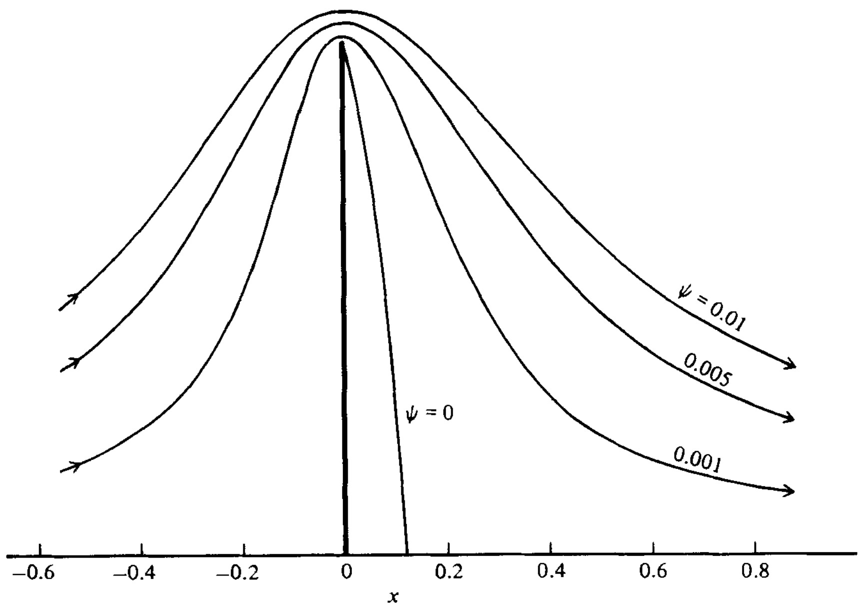

FIGURE 3. Flow pattern around the disk at $R = 0.3$.

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